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Research Article

Applications of δ-Open Sets via Separation Axioms, Covering Properties, and Rough Set Models

Tareq M. Al-Shami [1,2] **and Muneerah Al Nuwairan** [3]

[1] *Department of Mathematics, Sana'a University, Sana'a, Yemen*
[2] *Jadara University Research Center, Jadara University, Irbid, Jordan*
[3] *Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia*

Correspondence should be addressed to Muneerah Al Nuwairan; msalnuwairan@kfu.edu.sa

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In this article, we make use of δ-open sets to establish some topological concepts related to separation axioms and covering properties and to propose novel topological rough set models. We first demonstrate that the classes of regular-open and δ-open subsets of a finite topological space are equivalent when this space has the property of $\partial(A) \cap \partial(B) \subseteq \partial(A \cup B)$ for subsets A, B, where ∂ means the boundary topological operator. Then, we prove the equivalence between δT_i and T_i in the cases of $i = 2, 2(1/2)$ and construct an illustrative example showing that δ-topology is a proper class of an original topology in these two cases.With the help of counterexamples, we demonstrate that a normal (resp., a regular) space is a stronger condition than δ-normal (resp., δ-regular) and show that every T_i-space is δT_i for each $i = 3, 4$. We also reveal the relationships between the covering properties studied herein (compactness, Lindelöf, local compactness, and local Lindelöf) and investigate some equivalences when the topological space is δT_2. Furthermore, we discuss how the introduced concepts behave in the spaces of product topology, box topology, and sum topology. Finally, we set up a new framework for rough set models based on δ-open sets and examine its advantages and limitations in terms of retaining the characterizations of lower and upper approximations and increasing accuracy measures. An algorithm is designed to determine whether a subset is δ-definable or δ-rough.

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1. Introduction

In classical topology, repeated applications of closure and interior operators produce many diverse novel classes of sets. Some of them expand the class of open sets such as b-open and β-open sets [1], while few others shrink the class of open sets such as regular-open, δ-open, and θ-open sets. These classes of subsets represent one of the significant research directions in general topology, as they provide researchers with novel frameworks for studying classical topological principles and related notions, such as topological operators [2, 3], separation axioms [4, 5], compactness and covering properties [6–8], connectedness [9, 10], and continuity [11, 12]. Also, some of these classes have been applied to examine whether original topologies have certain properties or not; to name a few, the next properties:

- A topological space (U, ϑ) is hyperconnected iff a topological space (U, ϑ^δ) is indiscrete, where ϑ^δ is a topology generated by δ-open sets in ϑ.
- A topological space (U, ϑ) is connected iff a topological space (U, ϑ_θ) is connected, where ϑ_θ is a topology generated by θ-open sets in ϑ.

- A topological space (U, ϑ) is regular iff $\vartheta = \vartheta_\theta$.
- Every singleton subset of a topological space (U, ϑ) is open or pre-closed.

In practical applications, researchers have increasingly utilized these classes of open sets to address practical paradigms of information systems. For instance, Al-shami [13, 14] demonstrated how we can employ somewhat open sets and $\theta\beta$-open sets to describe information systems and offered a new instrument to deal with rough approximation operators. Salama [15] introduced new topological rough set models inspired by some generalizations of open sets, such as semi-open and regular-open sets and applied them to determine infection with dengue. Due to these applications, discussions of these classes have gained momentum in the last few years. In this regard, there are studies that highlight the importance of topological structures and motivate further investigation into their fundamental properties. For instance, Kaur et al. [16] applied nano-topological spaces with multi-ideals to diagnose and cure dengue. Yan and Yu [17] suggested a model of topological spatial relations for formal verification of geographic information systems in Coq.

The current work investigates new ideas and applications, all of which are defined by the class of δ-open subsets, introduced by Velichko [18] in 1966. In [18], Velichko first familiarized the concept of δ-cluster point and applied to establish the class of δ-closed sets and then he defined a δ-open set as the complement of a δ-closed set. In 1980, Noiri [19] discussed the concept of δ-continuity and revealed its relationships with some types of continuity, such as δ-continuity and strong δ-continuity. Afterward, many studies applied δ-open and δ-closed sets to initiate new forms of topological notions and examine their properties. Among these topological notions are semi-regularization topologies [20], extremally disconnectedness [21], minimal and maximal topologies [22], selection properties [23], primal topologies [24], and continuity [25, 26]. Recently, Roy et al. [27] have provided an equivalent definition for a δ-open set by using regular-open sets; that is, a set in a topological space is known as δ-open if it can be expressed as a union of regular-open sets. It is well known that the core idea of defining δ-open set is the regular-open set which was introduced by Stone [28] in 1937. He showed that the class of regular-open sets in any topological space constitutes a complete Boolean algebra; the join operation is given by $E \vee F = \text{int}(cl(E \cup F))$, the meet is $E \wedge F = E \cap F$, and the complement is $\neg E = \text{int}(E^c)$. Regular-open and regular-closed subsets have been exploited to investigate some topological ideas, such as topological operators [29] and covering properties [21].

The main contributions of this work are to describe classical topological concepts using the class of δ-open sets and to explore their key features. This work studies strong forms of separation axioms and covering properties and provides interesting examples showing the relationships between the concepts introduced herein and their counterparts in original topological spaces. It is well known that studying topological concepts through classes that are subsets of the open sets class is more challenging than studying them through classes that are supersets of the open sets class. However, we successfully reveal the interrelations between the proposed concepts and construct counterexamples to demonstrate the invalidity of certain implementations. Additionally, we put forward a new rough set model using δ-open sets and demonstrate the advantages of its induced rough approximation operators over those generated by regular-open and semi-open sets. As we prove, the proposed rough set paradigms keep all features of Pawlak's rough set model without needing extra requirements, which reflects the convergence of rough set theory and topology.

This paper is structured as follows. Section 2 reviews the definitions and results that make this manuscript self-contained as well as points out under which condition the regular-open and δ-open sets are equivalent and expands some previous results. In Section 3, we investigate the main properties of δT_i and build an elucidative example to show that δ-topology is a proper subset of an original topology in the cases of $i = 2, 2(1/2)$ in spite of the equivalence between δT_i and T_i for $i = 2, 2(1/2)$. Afterward, in Section 4, we look at the notions of compactness and Lindelöfness inspired by δ-open sets and discuss the condition under which we obtain some equivalences of these notions. In Section 5, we debate how the introduced concepts behave in the spaces of product topology, box topology, and sum topology. In Section 6, we exemplify the significance of the class of δ-open sets through an instance that illustrates how it can be used to preserve the properties of the original rough set model introduced by Pawlak and then design an algorithm to determine whether a subset is δ-definable or δ-rough. Ultimately, Section 7 concludes the manuscript with a summary and suggestions for further work.

2. Preliminaries

In this section, we review some basic definitions and results that are needed later. We also improve some previous results. Through this content, (U, ϑ) and (V, θ) refer to topological spaces.

Definition 1 (see [28]). An open subset A of (U, ϑ) is named a regular-open set if $A = \text{int}(cl(A))$. Its complement is named a regular closed set; that is, $A = cl(\text{int}(A))$.

Definition 2 (see [18]). In a topological space (U, ϑ), a point u is called a δ-cluster point of $A \subseteq U$ provided that for every open subset G of U containing u we have $A \cap \text{int}(cl(G)) \neq \varnothing$. The set of all δ-cluster points of A, denoted by $cl_\delta(A)$, is called the δ-closure of A. We call a subset A a δ-closed subset if $A = cl_\delta(A)$, and its complement is called a δ-open set.

In [27], it was proved the following result which provides an equivalent condition for a δ-open set.

Theorem 1. *A subset A of (U, ϑ) is a δ-open set iff for each point $u \in A$ there exists a regular-open set H satisfying that $u \in H \subseteq A$.*

That is, A is a δ-open set if and only if it equals the unions of all regular-open subsets of it. Consequentially, A is a δ-closed set if and only if it equals the intersections of all regular closed supersets of it.

Theorem 2 (see [19]). *The class of δ-open subsets of (U, ϑ) constitutes a topology on U, denoted by ϑ^{δ}.*

It is clear that $\vartheta^{\delta} \subseteq \vartheta$.

Theorem 3 (see [27]). *If A and B are, respectively, δ-open and open subsets of (U, ϑ), then $A \cap B$ is a δ-open subset of (B, ϑ_B).*

Definition 3 (see [27]). In a topological space (U, ϑ), a set $A \subseteq U$ is named:

i. δ-compact provided that there is a finite subcover for every cover of A by δ-open subsets of U.
ii. Locally δ-compact provided that for every $u \in A$ there exist a δ-open subset G and a δ-compact subset F of U such that $u \in G \subseteq F$; that is, every point of A has a δ-compact neighbourhood.

Theorem 4 (see [27])

i. A δ-closed set in a δ-compact space is δ-compact.
ii. The classes of closed and δ-closed subsets of a regular space coincide.
iii. The inverse image of a δ-open (resp., regular-open) set under a bicontinuous mapping is also δ-open (resp., regular-open).

Remember that a mapping $f\colon (U, \vartheta) \longrightarrow (V, \theta)$ is called bicontinuous [30] if it is continuous and open.

Definition 4 (see [31]). A topological space (U, ϑ) is said to be:

i. δT_0 if for every $v \neq w \in U$, there exists a δ-open subset that contains v but not w, or contains w but not v.
ii. δT_1 if for every $v \neq w \in U$, there exists a δ-open subset that contains v but not w, and a δ-open subset contains w but not v.
iii. δT_2 (or δ-Hausdorff) if for every $v \neq w \in U$, there exist disjoint δ-open subsets E and F containing v and w, respectively.

Proposition 1 (see [31])

i. Every δT_i-space is T_i for $i = 0, 1$.
ii. (U, ϑ) is T_2 iff it is δT_2.

Definition 5 (see [30]). Given two topologies ϑ_1 and ϑ_2 on a set U, ϑ_1 is said to be coarser (or weaker) than ϑ_2 if $\vartheta_1 \subseteq \vartheta_2$.

Definition 6 (see [22]). (U, ϑ) is said to be:

i. Minimal Hausdorff if ϑ is Hausdorff and there exists no Hausdorff topology on U strictly weaker than ϑ.
ii. Maximal compact if ϑ is compact and there exists no compact topology on U strictly stronger than ϑ.

Theorem 5 (see [32])

- *A topological space (U, ϑ) is maximal compact if and only if the compact and closed subsets of U are identical.*
- *If (U, ϑ) is compact Hausdorff, then it is both minimal Hausdorff and maximal compact.*

Definition 7 (see [19]). A mapping $f\colon (U, \vartheta) \longrightarrow (V, \theta)$ is said to be:

i. δ-continuous if the inverse image of each δ-open set is δ-open.
ii. δ-open if the image of each δ-open set is δ-open.

Definition 8 (see [33]). (U, ϑ) is said to be

i. Extremally disconnected if the closure of every open set is open.
ii. Hyperdisconnected if there exist disjoint nonempty open sets.

Definition 9 (see [34, 35]). A subset A of (U, ϑ) is named a pre-open (resp., semi-open) set if $A \subseteq \text{int}(cl(A))$ (resp., $A \subseteq cl(\text{int}(A))$). The complements of pre-open and semi-open sets are, respectively, named pre-closed and semi-closed sets.

Definition 10. (see [30]). If $\{(U_i, \vartheta_i)\colon i \in I\}$ is a family of pairwise disjoint topological spaces and $U = \cup_{i \in I} U_i$, then

$$\vartheta = \left\{G \subseteq \bigcup_{i \in I} U_i\colon G \cap U_i \text{ is an open set in } (U_i, \vartheta_i) \text{ for every } i \in I\right\}, \tag{1}$$

forms a topology on U, known as the sum topology.

In what follows, we explore under which condition the union of two regular-open sets is regular-open. We benefit from the next lemma that demonstrates the equivalent properties between interior and boundary operators.

Lemma 1 (see [36]). *Let A, B be subsets of (U, ϑ). Then the following properties are equivalent.*

i. $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$, and
ii. $\partial(A) \cap \partial(B) \subseteq \partial(A \cup B)$, where ∂ refers to a boundary operator.

Corollary 1. *Let A and B be regular-open subsets of (U, ϑ) such that $\partial(A) \cap \partial(B) \subseteq \partial(cl(A \cup B))$. Then $(A \cup B)$ is a regular-open set as well.*

Proof 1. Let A and B be subsets of (U, ϑ). Then, $cl(A \cup B) = cl(A) \cup cl(B)$. It is well known that $\partial(cl(A)) \subseteq \partial(A)$ for any subset A, so the given condition $\partial(A) \cap \partial(B) \subseteq \partial(cl(A) \cup cl(B))$ implies that $\partial(cl(A)) \cap \partial(cl(B)) \subseteq \partial(cl(A) \cup cl(B))$. By Lemma 1, we get that $\text{int}(cl(A \cup B)) = \text{int}(cl(A)) \cup \text{int}(cl(B))$. By hypothesis that A and B are regular-open subsets, we obtain $\text{int}(cl(A \cup B)) = A \cup B$, which proves that $A \cup B$ is a regular-open set. □

Corollary 2. *Let (U, ϑ) be a finite topological space such that $\partial(A) \cap \partial(B) \subseteq \partial(cl(A \cup B))$ for every regular-open subsets A, B of U. Then the concepts of regular-open and δ-open sets are equivalent.*

Proof 2. It is a direct result of Corollary 1. □

Result 1 (see [27]). In (U, ϑ), if $A \in \vartheta$, then $\text{int}(cl(A))$ is regular-open.

We see that the condition $A \in \vartheta$ is superfluous, so we represent this result as follows.

Proposition 2. *Let A be a subset of (U, ϑ). Then, $\text{int}(cl(A))$ is regular-open.*

Proof 3. Similar to the proof of Result 1 of [27]. □

Corollary 3. *Let A be a subset of (U, ϑ). Then, $cl(\text{int}(A))$ is regular closed.*

Remark 1. Note that if A is semi-open, then $cl(A) = cl(\text{int}(A))$. Therefore, $cl(A)$ is regular closed which implies that $cl(A) = cl_\delta(A)$.

By Theorem 1, one can remark that the class of regular-open subsets of a topological space (U, ϑ) structures a base for a topology consisting of δ-open subsets of U; this topology is exactly ϑ^δ. By this note we obtain the following result.

Proposition 3. *A mapping $f: (U, \vartheta) \longrightarrow (V, \theta)$ is δ-continuous if the inverse image of each regular-open set is regular-open.*

Proof 4. Let E be a δ-open subset of V. Therefore, it can be expressed as a union of regular-open subsets B_i; that is, black $E = \cup_{i \in I} B_i$. Now, $f^{-1}(E) = f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I}(f^{-1}(B_i))$. By hypothesis, $f^{-1}(B_i)$ is a regular-open subset for each i, so $f^{-1}(E)$ is a δ-open subset of U. This ends the proof that f is δ-continuous. □

3. δ-Separation Axioms

In this segment, we define classical separation axioms in terms of δ-open and δ-closed sets and discuss their main properties as well as reveal the relationships between them and the original ones.

Definition 11. A topological space (U, ϑ) is said to be:

i. $\delta T_{(1/2)}$ if $\{v\}$ is δ-open or δ-closed for every $v \in U$.
ii. $\delta T_{2(1/2)}$ if for every $v \neq w \in U$, there exist δ-open subsets E and F containing v and w, respectively, such that $cl_\delta(E) \cap cl_\delta(F) = \varnothing$.
iii. δ-regular if for every δ-closed subset H and each $v \notin H$, there exist disjoint δ-open subsets E and F containing H and v, respectively.
iv. δ-normal if there exist disjoint δ-closed subsets H and K and there exist disjoint δ-open subsets E and F containing H and K, respectively.
v. δT_3 (resp. δT_4) if it is both δ-regular (resp., δ-normal) and δT_1.

Since the family of δ-open subsets of (U, ϑ) forms a topology on U, the characterizations of classical T_i-spaces, regular, and normal spaces remain valid for δT_i-spaces, δ-regular, and δ-normal spaces. That is, the following proposition can be proved using similar approaches from classical topology. Therefore, we present them without proof.

Proposition 4. *A topological space (U, ϑ) is:*

i. δT_2 iff $\{v\} = \cap_i \{E_i$: E_i is a δ-closed neighborhood of $v\}$ for every $v \in U$.
ii. δ-normal iff for each δ-closed subset H and each δ-open subset F containing H, there exists a δ-open subset E such that $H \subseteq E \subseteq cl_\delta(E) \subseteq F$.

Proposition 5. *The following are equivalent.*

i. (U, ϑ) is δ-regular;
ii. For each $v \in U$ such that $v \in E \in \vartheta^\delta$ there exists $F \in \vartheta^\delta$ such that $v \in F \subseteq cl_\delta(F) \subseteq E$.
iii. Every δ-open subset E can be expressed as

$$E = \cup \left\{F: F \in \vartheta^\delta \text{ and } cl_\delta(F) \subseteq E\right\}. \quad (2)$$

Proposition 6. *If (U, ϑ) is a δ-regular space such that U is finite, then (U, ϑ) is δ-normal.*

As the next example illustrates, the converse of Proposition 6 does not hold in general.

Example 1. Let $\vartheta = \{\varnothing, \{x\}, \{y\}, \{x, y\}, U\}$ be a topology on $U = \{x, y, z\}$. One can prove that $\vartheta = \vartheta^\delta$. Note that there do not exist disjoint nonempty δ-closed subsets, so (U, ϑ) is δ-normal. On the other hand, $\{z\}$ is a δ-closed set and $x \notin \{z\}$, but we cannot find disjoint δ-open subsets; one of them contains $\{z\}$ and the other contains $\{x\}$; therefore, (U, ϑ) is not δ-regular.

In the following, we explore relationships between ϑ and ϑ^δ under certain δ-separation axioms, supported by illustrative examples.

Proposition 7. *If (U, ϑ) is δT_0 such that U is not a singleton, then (U, ϑ) is dishyperconnected.*

Proof 5. It follows from the fact that (U, ϑ^δ) is indiscrete iff (U, ϑ) is hyperconnected. □

Lemma 2. *If E and F are disjoint open subsets, then $\mathrm{int}(cl(E))$ and $\mathrm{int}(cl(F))$ are disjoint too.*

Proof 6. Since E and F are disjoint open subsets, we have the following:

$$\begin{aligned} E \cap cl(F) = \varnothing &\Rightarrow E \cap \mathrm{int}(cl(F)) = \varnothing \\ &\Rightarrow cl(E) \cap \mathrm{int}(cl(F)) \\ &= \varnothing \Rightarrow \mathrm{int}(cl(E)) \cap \mathrm{int}(cl(F)) = \varnothing. \end{aligned} \tag{3}$$

□

Proposition 8

i. *Every $\delta T_{(1/2)}$-space is $T_{(1/2)}$.*
ii. *Every normal space is δ-normal.*
iii. *Every regular is δ-regular.*

Proof 7. The proof of (i) is straightforward.

To prove (ii), take arbitrary disjoint δ-closed subsets H and K, so they are closed. By hypothesis, there are disjoint open subsets E and F containing H and K, respectively. Obviously, $H \subseteq E \subseteq \mathrm{int}(cl(E))$ and $K \subseteq F \subseteq \mathrm{int}(cl(F))$. Now, $\mathrm{int}(cl(E))$ and $\mathrm{int}(cl(F))$ are δ-open sets (by Proposition 2) and disjoint (by Lemma 2). Hence, (U, ϑ) is δ-normal.

The proof of (iii) follows directly from (ii) of Theorem 4. □

The converse of results in Proposition 8 is not always correct as the next counterexample illustrates.

Example 2. Let ϑ be the co-finite topology on the set of real numbers $\mathbb{R}$. It is well known that $(\mathbb{R}, \vartheta)$ is T_1. In contrast, it can be checked that ϑ^δ is the indiscrete topology on $\mathbb{R}$. Hence, $(\mathbb{R}, \vartheta)$ is not δT_0. Moreover, it can be seen that $(\mathbb{R}, \vartheta)$ is neither normal nor regular despite being δ-normal and δ-regular.

Proposition 9. *(U, ϑ) is $T_{2(1/2)}$ iff it is $\delta T_{2(1/2)}$.*

Proof 8. If (U, ϑ) is $\delta T_{2(1/2)}$, then by Remark 1, it follows that $cl(A) = cl_\delta(A)$ when A is open. Therefore, (U, ϑ) is $T_{2(1/2)}$. Conversely, let (U, ϑ) be $T_{2(1/2)}$. Then, by Remark 1, $cl(E) \cap cl(F) = \varnothing \Leftrightarrow cl_\delta(E) \cap cl_\delta(F) = \varnothing$ for open subsets E and F. Now, $\mathrm{int}(cl(E))$ and $\mathrm{int}(cl(F))$ are δ-open and we have $\mathrm{int}(cl(E)) \subseteq cl(E) = cl_\delta(E)$ and $\mathrm{int}(cl(F)) \subseteq cl(F) = cl_\delta(F)$. Hence, (U, ϑ) is $\delta T_{2(1/2)}$, as required. □

Corollary 4. *If (U, ϑ) is regular, then the concepts of δT_i and T_i are equivalent for each i. Moreover, $\vartheta = \vartheta^\delta$.*

Corollary 5. *Every compact δT_2-space is T_4.*

Note that Proposition 9 and item (ii) of Proposition 1 do not imply that $\vartheta^\delta = \vartheta$. To ensure this important point, we provide the next example.

Example 3. Let ϑ be the usual topology on the set of real numbers $\mathbb{R}$. We prove that the family $\theta = \{A \subseteq \mathbb{R}\colon A = E \backslash F$ such that $E \in \vartheta$ and F is countable is a topology on $\mathbb{R}$. First, every member G of the usual topology is a member of θ since $G = G \backslash \varnothing$; that is, $\vartheta \subset \theta$, so $\mathbb{R}$ and $\varnothing$ belong to θ. Second, let $\{A_i\colon i \in I\} \subseteq \theta$, so there exist $E_i \in \vartheta$ and a countable set F_i such that $A_i = E_i \backslash F_i$ for each i. Now, $\cup_{i \in I} A_i = \cup_{i \in I} (E_i \backslash F_i) = \cup_{i \in I} E_i \backslash \cap_{i \in I} F_i$. Since $\cup_{i \in I} E_i \in \vartheta$ and $\cap_{i \in I} F_i$ is a countable set, we obtain $\cup_{i \in I} A_i \in \theta$. Finally, let $A_1, A_2 \in \theta$. Then, $A_1 \cap A_2 = (E_1 \backslash F_1) \cap (E_2 \backslash F_2) = (E_1 \cap E_2) \backslash (F_1 \cup F_2)$. Since $E_1 \cap E_2 \in \vartheta$ and $F_1 \cup F_2$ is a countable set, we get $A_1 \cap A_2 \in \theta$, as required. This ends the proof that $(\mathbb{R}, \theta)$ is a topological space. It can be observed the following points:

i. $(\mathbb{R}, \theta)$ is $T_{2(1/2)}$ since θ contains the usual topology ϑ.
ii. The set of irrational numbers $\mathbb{Q}^c$ is an open subset of θ since it can be expressed as $\mathbb{Q}^c = \mathbb{R} \backslash \mathbb{Q}$.
iii. The closure of every open subset of $\mathbb{Q}^c$, except for the empty set, is $\mathbb{R}$ which implies that all nonempty subsets of $\mathbb{Q}^c$ are not regular-open sets. So $\mathbb{Q}^c$ is not δ-open.

Hence, θ^δ is a proper subset of θ.

Proposition 10. *Every T_i-space (U, ϑ) is δT_i for each $i = 3, 4$.*

Proof 9. When $i = 3$, we obtain by (ii) of Theorem 4 that a T_3-space (U, ϑ) is δT_3. □

When $i = 4$, we obtain by (ii) of Proposition 8 that a normal space (U, ϑ) is δ-normal. Now, since every T_4-space is T_2, it yields from (ii) of Proposition 1 that (U, ϑ) is δT_2. So that, (U, ϑ) is δT_1. Hence, (U, ϑ) is δT_4, as required.

Example 3 shows that the converse of Proposition 10 need not be true in general.

Now, we investigate which condition guarantees that $\vartheta^\delta = \vartheta$.

Proposition 11. *If (U, ϑ) is minimal T_2, then $\vartheta = \vartheta^\delta$.*

Proof 10. In general, we have $\vartheta^\delta \subseteq \vartheta$. According to (ii) of Proposition 1, we have (U, ϑ) is T_2 iff (U, ϑ^δ) is T_2. Since ϑ is minimal T_2, we obtain the equality between ϑ and ϑ^δ. □

Proposition 12. *If (U, ϑ) is regular, then $\vartheta = \vartheta^\delta$.*

Proof 11. The proof is warranted by (ii) of Theorem 4. □

The converse of Propositions 11 and 12 is generally false as the next counterexamples articulate.

Example 4. Let $\vartheta = \{\emptyset, \{x\}, \{y, z\}, U\}$ be a topology on $U = \{x, y, z\}$. Obviously, (U, ϑ) is not T_0. In contrast, it can be checked that $\vartheta = \vartheta^{\delta}$.

Example 5. Let ϑ be the double origin topology on $\mathbb{R} \cup \{x\}$, where $\mathbb{R}$ is the set of real numbers and $x \notin \mathbb{R}$. We know that the base of ϑ is regular-open subset, so every open subset of $(\mathbb{R} \cup \{x\}, \vartheta)$ is δ-open. Thus, $\vartheta = \vartheta^{\delta}$. In contrast, $(\mathbb{R} \cup \{x\}, \vartheta)$ is not regular.

Remark 2. According to Example 1, we show that $\vartheta = \vartheta^{\delta}$. But (U, ϑ) is not regular.

Proposition 13. *Let (U, ϑ) be extremely disconnected. Then, (U, ϑ) is regular iff $\vartheta = \vartheta^{\delta}$.*

Proof 12. The necessary part follows by the previous discussion. To prove the sufficiency, let G be an arbitrary open subset. Then, G is δ-open. Therefore, it can be written as a union of regular-open sets; that is, $G = \cup_{i \in I} A_i = \cup_{i \in I} \text{int}(cl A_i))$. By hypothesis of extremally disconnected, we get $G = \cup_{i \in I} cl(A_i)$. This implies that for each $u \in G$ there exists an open subset A_i such that $u \in A_i \subseteq cl(A_i) \subseteq G$. Hence, (U, ϑ) is regular. □

4. Covering Properties Generated by δ-Open Sets

In what follows, we discuss some properties of compact and Lindelöf spaces in terms of δ-open set. First, we show the interrelations between δ-compactness and local δ-compactness with the help of examples, as well as determine under which condition(s) they are equivalent.

Proposition 14

i. Every compact space is δ-compact.

ii. Every δ-compact space is locally δ-compact.

Proof 13. Straightforward. □

The converse of results given in Proposition 14 fails in general as we point out in the following example.

Example 6. Let ϑ be the particular point topology on the set of real numbers $\mathbb{R}$. It is well known that $(\mathbb{R}, \vartheta)$ is not compact. In contrast, it can be checked that ϑ^{δ} is the indiscrete topology on $\mathbb{R}$. Hence, $(\mathbb{R}, \vartheta)$ is δ-compact.

Example 7. Let ϑ be the usual topology on the set of real numbers $\mathbb{R}$. It is well known that $(\mathbb{R}, \vartheta)$ is regular, so by Proposition 12,we have $\vartheta = \vartheta^{\delta}$ which means that $(\mathbb{R}, \vartheta)$ is not δ-compact. On the other hand, $(\mathbb{R}, \vartheta)$ is locally δ-compact.

Proposition 15. *Let (U, ϑ) be T_2. Then,*

i. (U, ϑ) is compact iff it is δ-compact.

ii. (U, ϑ) is locally compact iff it is locally δ-compact.

Proof 14

i. The side "compactness implies δ-compactness" is given in Proposition 14. To prove the converse side, let (U, ϑ) be δ-compact T_2; then by Proposition 12, (U, ϑ) is δ-compact δT_2. Therefore, (U, ϑ) is regular which implies that $\vartheta = \vartheta^{\delta}$. Hence, (U, ϑ) is compact, as required.

ii. Necessity: Let (U, ϑ) be locally compact. Then, for every $u \in U$ there exists an open subset G and a compact subset F of U such that $u \in G \subseteq F$. By hypothesis of T_2, F is closed, so $u \in \text{int}(cl(G)) \subseteq F$. Now, $\text{int}(cl(G))$ is δ-open and F is δ-compact; hence, (U, ϑ) is locally δ-compact. By the equivalence given in (i) above, and the fact that every δ-open is open, the sufficient part is proved.

Note that Example 7 demonstrates that a locally δ-compact space is not δ-compact even if the topological space is Hausdorff. □

Proposition 16. *If B is an open and δ-closed subset of a locally δ-compact space (U, ϑ), then (B, ϑ_B) is locally δ-compact.*

Proof 15. Let (U, ϑ) be locally δ-compact and $b \in B \subseteq U$. Then, there exists a δ-open subset G and a δ-compact subset F of U such that $b \in G \subseteq F$, so $b \in G \cap B \subseteq F \cap B$. By Theorem 3, $G \cap B$ is a δ-open subset of (B, ϑ_B). Now, $F \cap B$ is a δ-compact subset of (U, ϑ), so it is a δ-compact subset of (B, ϑ_B) as well. Hence, we prove that (B, ϑ_B) is locally δ-compact. □

To expand the previous covering properties, we furnish the concepts of δ-Lindelöf and locally δ-Lindelöf spaces.

Definition 12. A subset A of (U, ϑ) is called:

i. δ-Lindelöf provided that there exists a countable subcover for any open cover of A by δ-open subsets of U.

ii. Locally δ-Lindelöf provided that for every $u \in A$ there exist δ-open subset G and δ-Lindelöf subset F of U such that $u \in G \subseteq F$; that is, every point of A has a δ-Lindelöf neighbourhood.

I am not sure if there is a version of locally δ-Lindelöf defined by open and Lindelöf subsets in the literature review. So, we give its definition as follows.

Definition 13. A subset A of (U, ϑ) is called locally Lindelöf provided that for every $u \in A$ there exist an open subset G and a Lindelöf subset F of U such that $u \in G \subseteq F$; that is, every point of A has a Lindelöf neighbourhood.

One can easily note that a locally compact space is locally Lindelöf, but the converse is not true in general, as we show in the following example.

Example 8. Let ϑ be the complement countable topology on the set of real numbers $\mathbb{R}$. It can be seen that $(\mathbb{R}, \vartheta)$ is locally Lindelöf because it is Lindelöf itself. In contrast, this topological space is not locally compact.

It is clear that every δ-compact (resp., locally δ-compact) space is δ-Lindelöf (resp., locally δ-Lindelöf), whereas any discrete topology on infinite countable set is δ-Lindelöf but not δ-compact. Also, double origin topology is second countable, so it is locally δ-Lindelöf. In contrast, it is T_2 but not locally compact; hence, it is not locally δ-compact by (ii) of Proposition 15.

The results obtained in Proposition 14 are still valid for Lindelöfness. Example 6 and the next one elucidate that the converse of these results need not be true in the case of Lindelöfness too.

Example 9. It is known that every order topological space is T_3. Take the order topology ϑ on $[0, \omega_1)$. It can be checked that $([0, \omega_1), \vartheta)$ is not δ-Lindelöf. On the other hand, every order topological space is locally δ-compact, so it is locally δ-Lindelöf.

It worth noting that a δ-Lindelöf δT_2-space sometimes fails to be a δ-regular space as shown by a double origin topological space.

Proposition 17. *Every regular Lindelöf space is δ-normal.*

Proof 16. Since $\vartheta = \vartheta^{\delta}$ under the condition of regularity, the result is proved by following a similar approach of its analogous result in general topology. □

To illustrate how δ-open sets transition between topologies related by inclusion, we first provide examples showing that no relationship exists, in general. Then, we demonstrate a sufficient condition that guarantees this transition and use it to study δ-compact and δ-Lindelöf spaces within these topologies.

Example 10. Let

$$\begin{aligned} \vartheta_1 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \mathbb{R}\}, \\ \vartheta_2 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \mathbb{R}\backslash\{2\}, \mathbb{R}\} \end{aligned} \tag{4}$$

be topologies on the set of real numbers $\mathbb{R}$. Note that a subset $\{1, 3\}$ is a δ-open subset of $(\mathbb{R}, \vartheta_1)$, but it is not a δ-open subset of $(\mathbb{R}, \vartheta_2)$.

Example 11. Let $\vartheta_1 = \{\emptyset, \{1\}, \mathbb{R}\}$ be a topology on the set of real numbers $\mathbb{R}$ and ϑ_2 be the discrete topology on $\mathbb{R}$. Obviously, $\{1\}$ is a δ-open subset of $(\mathbb{R}, \vartheta_2)$, but it is not a δ-open subset of $(\mathbb{R}, \vartheta_1)$.

Theorem 6. *Let ϑ_1 and ϑ_2 be topologies on U such that $\vartheta_1 \subseteq \vartheta_2$ and ϑ_1 is extremally disconnected. If A is a δ-open subset of (U, ϑ_1), then it is a δ-open subset of (U, ϑ_2).*

Proof 17. Let A be a δ-open subset of (U, ϑ_1). Since ϑ_1 is extremally disconnected, $A = \mathrm{int}_{\vartheta_1}(cl_{\vartheta_1}(A)) = cl_{\vartheta_1}(A)$. Therefore, A is a clopen subset of (U, ϑ_1). Since $\vartheta_1 \subseteq \vartheta_2$, we obtain A is a clopen subset of (U, ϑ_2), which means that A is a δ-open subset of (U, ϑ_2). □

Corollary 6. *Let ϑ_1 and ϑ_2 be topologies on U such that $\vartheta_1 \subseteq \vartheta_2$ and ϑ_1 is extremally disconnected. If (U, ϑ_2) is δ-compact (resp., δ-Lindelöf), then (U, ϑ_1) is δ-compact (resp., δ-Lindelöf).*

The converse of the above proposition and corollary is not true as the next example shows.

Example 12. It can be seen from Example 11 that $(\mathbb{R}, \vartheta_1)$ is extremally disconnected and δ-compact, whereas $(\mathbb{R}, \vartheta_2)$ is not δ-Lindelöf.

Now, we investigate the strong types of the previous ones.

Definition 14. A subset A of (U, ϑ) is called strongly locally δ-compact (resp., strongly locally δ-Lindelöf) provided that for every $u \in A$ there exists a δ-open subset G containing u such that $cl_{\delta}(G)$ is δ-compact (resp., δ-Lindelöf).

Proposition 18. *Every strongly locally compact (resp., strongly locally Lindelöf) is strongly locally δ-compact (resp., strongly locally δ-Lindelöf).*

Proof 18. Let (U, ϑ) be strongly locally compact. Then, for every $u \in U$ there exists an open subset G containing u such that $cl(G)$ is compact. Now, $\mathrm{int}(cl(G))$ is a δ-open subset containing u. Since $cl(\mathrm{int}(cl(G)))$ is a closed subset of a compact set $cl(G)$, we obtain that $cl(\mathrm{int}(cl(G)))$ is also a compact subset. So, it is δ-compact. Hence, we obtain a δ-open subset $\mathrm{int}(cl(G))$ and a δ-compact subset $cl(\mathrm{int}(cl(G)))$ such that $u \in \mathrm{int}(cl(G)) \subseteq cl(\mathrm{int}(cl(G)))$, which ends the proof that (U, ϑ) is strongly δ-locally compact. □

By following a similar argument, one can prove the case between parentheses.

Corollary 7. *Every strongly locally δ-compact (resp., strongly locally δ-Lindelöf) is locally δ-compact (resp., locally δ-Lindelöf).*

In Example 6, we proved that $(\mathbb{R}, \vartheta^{\delta})$ is δ-compact. So, it is strongly locally δ-compact. On the other hand, it can be seen that the closure of any open subset is $\mathbb{R}$, which is not Lindelöf. This implies that we cannot find an open subset G containing the particular point such that $cl(G)$ is Lindelöf.

This shows that the converse of Proposition 18 is generally false.

5. Behaviors of δ-Separation Axioms and δ-Covering Properties via Some Topologies

In the rest of this section, we shall investigate how the previous concepts behave under some topologies that are generated from old ones such as product, box, and sum topologies.

Remark 3. It is well known in the product spaces $(\prod_{i\in I}U_i, T)$ of general topology that $cl(\prod_{i\in I}A_i) = \prod_{i\in I}cl(A_i)$ and $\text{int}(\prod_{i\in I}A_i) \subseteq \prod_{i\in I}\text{int}(A_i)$. The equality $\text{int}(\prod_{i\in I}A_i) = \prod_{i\in I}\text{int}(A_i)$ holds in case of I is finite or $A_i = X_i$ for all but finitely many i. According to these facts, we conclude that the finite product of δ-open sets is δ-open, whereas this property need not be true under infinite product of spaces. On the other hand, both distributive properties $cl(\prod_{i\in I}A_i) = \prod_{i\in I}cl(A_i)$ and $\text{int}(\prod_{i\in I}A_i) = \prod_{i\in I}\text{int}(A_i)$ hold true under box topological space even if I is infinite.

By the above remark, we have the next results.

Proposition 19

i. A subset $\prod_{i\in I}A_i$ of the finite product of spaces $(\prod_{i\in I}U_i, T)$ is δ-open iff each A_i is a δ-open subset of (U_i, τ_i).

ii. A subset $\prod_{i\in I}A_i$ of the box topological space $(\prod_{i\in I}U_i, T)$ is δ-open iff each A_i is a δ-open subset of (U_i, τ_i).

Item (i) of the above proposition is not true in the case of an infinite index set as articulated in the following example.

Example 13. Let $(\mathbb{R}_n, \vartheta_n)$ be the usual topological space for each $n \in \mathbb{N}$ and let $(\prod_{n\in\mathbb{N}}\mathbb{R}, T)$ be the product of these topological spaces, where $\mathbb{N}$ is the set of natural numbers. By taking the open interval set $(1, 3)$, we obtain that

$$\begin{aligned}&\text{int}(cl((1,3)\times(1,3)\times\cdots\times(1,3)\times\cdots)) = \text{int}([1,3]\times[1,3]\times\cdots\times[1,3]\times\cdots) = \varnothing, \text{ whereas}\\ &\text{int}(cl(1,3))\times\text{int}(cl(1,3))\times\cdots\times\text{int}(cl(1,3))\times\cdots = (1,3)\times(1,3)\times\cdots\times(1,3)\times\cdots.\end{aligned} \tag{5}$$

Proposition 20

i. The product topological space is a δT_i-space iff each one of the topological spaces is δT_i, where $i = 0, (1/2), 1, 2, 2(1/2), 3$.

ii. The box topological space is a δT_i-space iff each one of the topological spaces is δT_i, where $i = 0, (1/2), 1, 2, 2(1/2), 3$.

Proof 19

i. In the cases of $i = 2, 2(1/2)$, the proof follows from the equivalent between δT_i and their classical ones in general topology. And the proof of the cases of $i = 0, (1/2), 1$ is straightforward.

ii. It follows from (ii) of Proposition 19. □

Proposition 21

i. The product topological space (finite box topological space) is δ-compact iff each one of the topological spaces is δ-compact.

ii. The finite product topological space is locally δ-compact iff each one of the topological spaces is locally δ-compact.

iii. The box topological space is locally δ-compact iff each one of the topological spaces is locally δ-compact

Proof 20. It can be obtained from (ii) of Proposition 10 and the analogous results in general topology. □

It should be noted that if the index set is infinite, then the box topological space need not be δ-compact even if each one of the topological spaces is δ-compact.

The aforementioned proposition is not true in the case of δ-Lindelöf as shown by classical example of lower or upper limit topology (Sorgenfrey plane). In [37], it was proved that Sorgenfrey plane is not normal. On the other hand, it is δT_3. Hence, this example also demonstrates that the product of δ-normal spaces is not always δ-normal, which automatically means that the product of δT_4-spaces is not δT_4 in general.

Remark 4. Topologists prefer product topologies since in contrast to the box topology, it preserves important properties such as compactness and connectedness.

It remains to look at how the previous concepts inspired by δ-open sets behave in the spaces of sum topology.

We need the following lemma.

Lemma 3. *Let A be a subset of a sum topological space $(\oplus_{i\in I}U_i, \oplus_{i\in I}\vartheta)$. Then,*

i. $\text{int}(A) = \cup_{i\in I}\text{int}_i(A\cap U_i)$.

ii. $cl(A) = \cup_{i\in I}cl_i(A\cap U_i)$.

Proof 21. Straightforward. □

Proposition 22. *A subset A of a sum topological space* $(\oplus_{i\in I}U_i, \oplus_{i\in I}\vartheta)$ *is* δ*-open (resp.,* δ*-closed) if and only if* $A\cap U_i$ *is* δ*-open (resp.,* δ*-closed) for each i.*

Proof 22. $\Rightarrow$: Since A is a δ-open subset of $(\oplus_{i\in I}U_i, \oplus_{i\in I}\vartheta)$, $\text{int}(cl(A)) = A = \cup_{i\in I}(A\cap U_i)$. This means that

$$\text{int}_i\Big(cl_i\Big(\underset{i\in I}{\cup} A\cap U_i\Big)\Big) = \underset{i\in I}{\cup}(A\cap U_i). \tag{6}$$

By the definition of sum topology, $U_i\cap Uj = \varnothing$ for each $i\neq j$, so the family $\{A\cap U_i\colon i\in I\}$ is pairwise disjoint; thus, $\text{int}_i(cl_i(\cup_{i\in I}A\cap U_i)) = \cup_{i\in I}\text{int}_i(cl_i(A\cap U_i))$ for each i. By pairwise disjointness of $\{\text{int}_i(cl_i(A\cap U_i))\colon i\in I\}$, we get from (1) that $\text{int}_i(cl_i(A\cap U_i)) = (A\cap U_i)$ for each i.

$\Leftarrow$: Let $A\cap U_i$ be a δ-open subset of (U_i, ϑ_i) for each i; that is, $\text{int}_i(cl_i(A\cap U_i)) = (A\cap U_i)$ for each i. This automatically leads to that $\cup_{i\in I}\text{int}_i(cl_i(A\cap U_i)) = \cup_{i\in I}(A\cap U_i)$. By pairwise disjointness of $\{\text{int}_i(cl_i(A\cap U_i))\colon i\in I\}$, we get $\text{int}_i(cl_i(\cup_{i\in I}(A\cap U_i))) = \cup_{i\in I}(A\cap U_i)$. Again, by (1), we obtain $\text{int}(cl(A)) = A$, which finishes the proof. □

Proposition 23. *A sum topological space* $(\oplus_{i\in I}U_i, \oplus_{i\in I}\vartheta)$ *is*

- *i.* δT_i *if and only if* (U_i, ϑ_i) *is* δT_i *for each i.*
- *ii.* δ*-compact (resp.,* δ*-Lindelöf) if and only if I is finite (resp., countable) and* (U_i, ϑ_i) *is* δ*-compact (resp.,* δ*-Lindelöf) for each i.*
- *iii. Locally* δ*-compact (resp., locally* δ*-Lindelöf) if and only if* (U_i, ϑ_i) *is locally* δ*-compact (resp., locally* δ*-Lindelöf) for each i.*
- *iv. Strongly locally* δ*-compact (resp., strongly locally* δ*-Lindelöf) if and only if* (U_i, ϑ_i) *is strongly locally* δ*-compact (resp., strongly locally* δ*-Lindelöf) for each i.*

6. An Application of δ-Open Sets in Generating Information Systems

In this part, we shall make use of the class of δ-open sets to create new rough set models. To be familiar with this topic, we recall the definition of right neighbourhood (briefly, r-neighbourhood).

Definition 15. Let ρ be a binary relation on a nonempty set U. The r-neighbourhood of an element $u\in U$ is given by

$$\mathcal{N}_r(u) = \{v\in U\colon (u,v)\in\rho\}. \tag{7}$$

Definition 15 was exploited to initiate a topology on U as follows (for more details, see [38]):

$$\mathfrak{B} = \{\mathcal{N}_r(u)\colon \text{ for each } u\in U\} \text{ is a subbase for a topology on } U. \tag{8}$$

Due to the close similarity between topological operators (interior and closure) and rough approximation operators (lower and upper), topology, built by (8), has been applied to describe rough set models using topological concepts. The confirmed and possible knowledge obtained from given set of data A is known in rough set theory as lower approximation $L(A)$ and upper approximation $T(A)$, respectively. We measure these approximations using interior and closure topological operators, which represents their topological counterparts; that is, $L(A) = \text{int}(A)$ and $T(A) = cl(A)$.

To extend the applicability of topological methods in rough set theory, researchers have utilized superclasses and subclasses of topology to characterize rough approximation operators. One of the desirable characteristics of these models is to preserve the distributive properties of union and intersection of subsets under lower and upper rough approximations, which is due to the fact that the class of δ-open sets constructs a topology. As a matter of fact, these properties are missing for rough set models inspired by most generalizations of open sets like semi-open and pre-open sets, or by some subclasses of open sets like regular-open sets. On the other hand, the accuracy measure computed from the proposed models decreases for some subsets compared to the existing one. However, the proposed models will be better than some other models inspired by some classes of sets such as clopen and regular-open subsets. Since the topology produced by the Pawlak model is a clopen topology, using open sets in granular computing is not different from using δ-open sets in such a model.

To illustrate these facts, we provide the next example.

Example 14. Let $\rho = \{(v,v), (w,w), (x,v), (x,w)\}$ be a binary relation on $U = \{v,w,x\}$. Then, by (2), we have $\mathcal{N}_r(v) = \{v\}$, $\mathcal{N}_r(w) = \{w\}$, and $\mathcal{N}_r(x) = \{v,w\}$. According to (3), $\vartheta = \{\varnothing, \{v\}, \{w\}, \{v,w\}, U\}$ is a topology on U. Now, we explore rough set models inspired by the classes of clopen, regular-open, δ-open, and semi-open subsets of (U,ϑ). Theses classes are, respectively, given in the following:

$$\begin{aligned} &\text{class of clopen subsets CO}(U,\vartheta) = \{\varnothing, U\},\\ &\text{class of regular–open subsets RO}(\mathrm{U},\vartheta) = \{\varnothing, \{v\}, \{w\}, U\},\\ &\text{class of } \delta\text{–open subsets } \delta\mathrm{O}(\mathrm{U},\vartheta) = \{\varnothing, \{v\}, \{w\}, \{v,w\}, U\} = \vartheta,\\ &\text{class of semi–open subsets SO}(\mathrm{U},\vartheta) = \vartheta\cup\{\{v,x\}, \{w,x\}\}. \end{aligned} \tag{9}$$

Table 1: Lower and upper approximations inspired by the classes of CO, RO, δO, and SO.

Classes	CO		RO		δO		SO	
$A \subseteq U$	$L_{co}(A)$	$T_{co}(A)$	$L_{ro}(A)$	$T_{ro}(A)$	$L_{\delta o}(A)$	$T_{\delta o}(A)$	$L_{so}(A)$	$T_{so}(A)$
$\{v\}$	$\varnothing$	U	$\{v\}$	$\{v,x\}$	$\{v\}$	$\{v,x\}$	$\{v\}$	$\{v\}$
$\{w\}$	$\varnothing$	U	$\{w\}$	$\{w,x\}$	$\{w\}$	$\{w,x\}$	$\{w\}$	$\{w\}$
$\{x\}$	$\varnothing$	U	$\varnothing$	$\{x\}$	$\varnothing$	$\{x\}$	$\varnothing$	$\{x\}$
$\{v,w\}$	$\varnothing$	U	$\{v,w\}$	U	$\{v,w\}$	U	$\{v,w\}$	U
$\{v,x\}$	$\varnothing$	U	$\{v\}$	$\{v,x\}$	$\{v\}$	$\{v,x\}$	$\{v,x\}$	$\{v,x\}$
$\{w,x\}$	$\varnothing$	U	$\{w\}$	$\{w,x\}$	$\{w\}$	$\{w,x\}$	$\{w,x\}$	$\{w,x\}$

```
Input: A binary relation ρ over the universal set U.
Output: Classify a subset as δ-definable or δ-rough.
Determine a binary relation ρ over U;
for all x ∈ U do
    Compute 𝒩_r(x)
end
Construct a topology ϑ on U by a formula given in formula (8);
for each set A in ϑ do
    Compute cl(A);
    Put cl(A) = B;
    Compute int(B)
end
Write ϑ^δ = {∅, U, A: A = int(B)};
for each set A in ϑ^δ do
    Compute int_δ(A);
    Put L_δo(A) = int_δ(A);
    Compute cl_δ(A);
    Put T_δo(A) = cl_δ(A);
    if L_δo(A) = T_δo(A) then
        return A is a δ-definable set
    else
        A is a δ-rough set
    end
end
```

Algorithm 1: Determination of whether a subset is δ-definable or δ-rough.

The lower and upper approximations are calculated with respect to these classes by using the following formulas:

$$
\begin{aligned}
L_{co}(A) &= \cup\{G \subseteq U: G \subseteq A \text{ and } G \in CO(U,\vartheta)\}, T_{co}(A) = \cap\{F \subseteq U: A \subseteq F \text{ and } F^c \in CO(U,\vartheta)\},\\
L_{ro}(A) &= \cup\{G \subseteq U: G \subseteq A \text{ and } G \in RO(U,\vartheta)\}, T_{ro}(A) = \cap\{F \subseteq U: A \subseteq F \text{ and } F^c \in RO(U,\vartheta)\},\\
L_{\delta o}(A) &= \cup\{G \subseteq U: G \subseteq A \text{ and } G \in \delta O(U,\vartheta)\}, T_{\delta o}(A) = \cap\{F \subseteq U: A \subseteq F \text{ and } F^c \in \delta O(U,\vartheta)\},\\
L_{so}(A) &= \cup\{G \subseteq U: G \subseteq A \text{ and } G \in SO(U,\vartheta)\}, T_{so}(A) = \cap\{F \subseteq U: A \subseteq F \text{ and } F^c \in SO(U,\vartheta)\}.
\end{aligned}
\tag{10}
$$

Now, we calculate the lower and upper approximations inspired by these classes in Table 1.

On the one hand, one can see from Table 1 that $L_{so}(\{v,x\} \cap \{w,x\}) = L_{so}(\{x\}) = \varnothing \subseteq L_{so}(\{v,x\}) \cap L_{so}(\{w,x\}) = \{x\}$ and $T_{so}(\{v\}) \cup T_{so}(\{w\}) = \{v,w\} \subseteq T_{so}(\{v,w\}) = U$. That is, the rough set models inspired by the class of SO do not satisfy the distributive properties of union and intersection of subsets under lower and upper rough approximations. But, this property is satisfied by the rough set models inspired by the class of δO. On the other hand, according to the computations given in Table 1, one can see that the confirmed knowledge obtained by the class of δO strictly lies between the classes of CO and SO; that is, the lower approximation generated by the class of δO is greater than its counterpart inspired by the class of CO and less than its counterpart inspired by the class of SO. We conclude from this discussion that the rough set models induced from the class of δO have two advantages: first, preserve the properties of the standard model of Pawlak, and second remove or reduce uncertainty more than some subclasses of open subsets such as clopen and θ-open subsets.

Remark 5. We draw the reader's attention to the fact that there are other formulas applied to describe rough approximation operators utilizing topological methods, such as those proposed by Salama and Abd El-Monsef [39] as follows:

$$\vartheta = \{A \subseteq U\colon \mathcal{N}_r(u) \subseteq A \text{ for each } u \in A\} \text{ is a topology on } U. \tag{11}$$

According to the binary relation given in Example 14, the topology generated by the method of (11) is the same as the topology given in Example 14 that is created by the method in (8). Thus, the classes of clopen, regular-open, δ-open, and semi-open subsets are not changed. Hence, these computations validate the aforementioned advantages of δ-open subsets in the topological characterization of rough set models.

Definition 16. A subset A of (U, ϑ) is said to be δ-definable if $L_{\delta o}(A) = T_{\delta o}(A)$. Otherwise, A is said to be a δ-rough set.

According to Example 14, all subsets are δ-rough.

We examine whether a subset is δ-definable or δ-rough following steps given in Algorithm 1.

7. Conclusions and Future Work

In topology, generalizations of open sets are important concepts that are routinely applied and have significant influence across diverse scientific and engineering domains. This article enriches the existing literature on novel topological properties by using a new category of open sets, namely, δ-open sets. This study provides new examples and characteristics, enhancing our overall comprehension of topological spaces. This paper delves into five research directions illustrated in the following:

i. Discuss the main properties of δT_i and explore the sufficient condition to obtain the identity between topology and δ-topology under some of these axioms.
ii. Study some covering properties inspired by δ-open sets such as δ-compact and δ-Lindelöf spaces.
iii. Elucidate how the given concepts of separation axioms and covering properties behave in product topology, box topology, and sum topology.
iv. Structure some illustrative counterexamples to illuminate the obtained relationships.
v. Illustrate how we can benefit from the class of δ-open subsets to analyze and describe approximation spaces of information systems. The convergence of rough set theory and topology has been demonstrated by proving the validity of all properties of Pawlak's rough set model within the framework defined by δ-open sets, without requiring additional conditions.

We draw the readers' attention to the fact that the topological space generated by δ-open sets does not yield new results for the fixed-point theorem. This is because the obtained results rely on the conditions of compactness and Hausdorffness, which we have proven to be sufficient to establish the equivalence between the topology and δ-topology. Even in terms of the size of the δ-topology, the conditions of compactness and Hausdorffness imply regularity, leading to the equality between the topology and δ-topology, as articulated in this study.

It is evident from the published literature that uncertain types of topology, such as fuzzy and soft topologies, are vital tools for addressing various challenges we encounter in different real-life situations. Therefore, in future research, we believe it would be beneficial to explore the proposed concepts in fuzzy and soft settings and investigate their potential applications.

Data Availability Statement

No data were used in the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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